Displacement Equations of Linear Elasticity in

Differential Form Notation

Frank Schadt
Pforzheim University of Applied Sciences, Germany
Dept. of Measurement and Optoelectronics
September 08

Introduction

Differential forms allow a modern, often easier and clearer approach to problems which are usually written in gibbsian vector calculus form. They provide a generalization of vector calculus expressions like rotation, gradient and divergence, of various integral theorems like Gauss’ divergence theorem or Stoke’s integral theorem on planes to just ONE integral theorem, and make coordinate transformations in curvilinear coordinate systems much clearer than the Riemannian geometry approach using connections for general orthogonal coordinate systems. For an introduction to differential forms please confer to [Fla90], [Hub02], [Ara].

Unfortunately it is not always possible to express vector or tensor equations as equations of differential forms. This is mainly because of the antisymmetric nature of differential forms of degree $\geq 2$. While e.g. the gradient of scalar functions and the rotation and divergence of vector fields can be expressed using differential forms, this is not possible with the gradient of a vector field (actually it is, but one has to use vector valued differential forms, which don’t simplify things much).

It is, however, possible to express certain differential equations of vector fields as ones of differential forms. In electrodynamics, differential forms are already gaining popularity as a “language” to express maxwell’s equations etc.[Sel]. In continuum mechanics, however, there are only few steps towards using differential forms. Maybe the most famous proponent is J. E. Marsden [Mar88], [Mar94] who published several books on the application of modern differential geometrical concepts in mechanics.

The aim of this paper is merely to provide an example of how to apply differential forms on the simplest displacement equation that occurs in elasticity: the static displacement equation of linear isotropic materials.

General Remarks

In this article, the Einstein summation convention is used, unless otherwise stated. If one of the indices is in parantheses (e.g. $a_i dx^i$), then no summation is applied.

The corresponding one-form of a vector field $^b u$ is assigned the same letter, but without the vector arrow. E.g.: $^b u = \tilde{g} \cdot \tilde{u} =: u$. Here, the superscript $b$ denotes an index lowering operation, $g$ is a metric tensor and $u$ is the corresponding one-form or covector field to the vector field $^b u$.

In this paper, not much attention is payed to topological or set theoretic issues. We always assume the euclidian 3-space $\mathbb{E}^3$ or bounded submanifolds of it as domain for our solution. Domain definitions shall serve mainly to inform, which type of vector, tensor or differential form is used.

Displacement Equation of Static Linear Elasticity

The basic differential equation of static linear elasticity for isotropic solids is a linear partial differential equation in the vector-valued displacement field $\tilde{u}(\tilde{x})$ [Tim70]:

$$\mu \Delta \tilde{u} + (\lambda + \mu) \text{grad} \text{div} \tilde{u} = 0$$

(1)

It is well known how to express the divergence and the rotation of vector fields as exterior derivatives of a differential form. The laplace operator, however, involves the calculation of

\footnote{the “correspondence” being the so-called musical isomorphism or index lowering / raising operation. cf. e.g. [Bis80]}
the gradient:
\[ \Delta \bar{u} = \text{div} \text{grad} \bar{u} \]  \hspace{1cm} (2)

The gradient of a vector field is a general second rank tensor and is thus not representable by a scalar-valued two-form. One possibility would be a vector valued one-form, but this offers no calculational benefit over the classical gradient.

It’s possible to express the laplace operator using only rot and div of vector fields and grad of a scalar field:
\[ \Delta \bar{u} = \text{grad} \text{div} \bar{u} - \text{rot} \text{rot} \bar{u} \]  \hspace{1cm} (3)

Now we express the displacement vector field \( \bar{u} \in TE^3 \) as a differential form
\[ u := u^{(p)} e^i = u^{(c)} dx^i \in \Omega^1 E^3 \equiv T^* E^3 \], either using physical components \( u^{(p)} \) together with an orthonormal covector basis \( \{ e^i \} \) or canonical components \( u^{(c)} \) or just \( u \) with the canonical basis \( \{ dx^i \} \).

The three operations grad, div and rot are in differential form notation as follows [Ara]:
\[ \text{grad} f \leftrightarrow df \in \Omega^1 E^3 \]  \hspace{1cm} (4)
\[ \text{div} \bar{u} \leftrightarrow d * u \in \Omega^1 E^3 \) or \( \text{div} \bar{u} \leftrightarrow \ast d * u \in \Omega^0 E^3 = R \)  \hspace{1cm} (5)
\[ \text{rot} \bar{u} \leftrightarrow du \in \Omega^2 E^3 \) or \( \text{rot} \bar{u} \leftrightarrow \ast d u \in \Omega^1 E^3 \)  \hspace{1cm} (6)

The double arrow \( \leftrightarrow \) is used without definition. It is used to express the possibility of replacement of the classical vector analysis operations grad, div and rot.

The summands in (3) can thus be reformulated as \( \text{grad} \text{div} \bar{u} \leftrightarrow d * d * u \in \Omega^1 E^3 \) and \( \text{rot} \text{rot} \bar{u} \leftrightarrow \ast d * \ast d u \in \Omega^1 E^3 \), respectively.

Now it’s possible to reformulate (1) using differential forms:
\[ \mu \Delta \bar{u} + (\lambda + \mu) \text{grad} \text{div} \bar{u} = (\lambda + 2\mu) \text{grad} \text{div} \bar{u} - \mu \text{rot} \text{rot} \bar{u} = 0 \]  \hspace{1cm} (7)

\[ (\lambda + 2\mu) d * d * u - \mu \ast d * du = 0 \]  \hspace{1cm} (8)

Or, using the codifferential operator \( \delta \), defined as follows:
\[ \delta : \Omega^n (T^* R^3) \rightarrow \Omega^{n-1} (T^* R^3) \]
\[ \omega \mapsto \left\{ \begin{array}{l} \ast d \ast \omega \text{ if } \omega \in \Omega^3 (T^* R^3) \text{ or } \omega \in \Omega^1 (T^* R^3) \\ \ast d \ast \omega \text{ if } \omega \in \Omega^2 (T^* R^3) \end{array} \right\} \]  \hspace{1cm} (9)

\[ (\lambda + 2\mu) d \delta u + \mu \delta du = 0 \]  \hspace{1cm} (10)

**From Vector Fields to One-Forms**

**The Musical Isomorphism**

The “musical isomorphism flat” or index lowering operation map is defined as follows:
\[ b : TM \rightarrow T^* M \]
\[ \bar{v} \mapsto \bar{v}^b = v = \bar{g} \cdot \bar{v} \]  \hspace{1cm} (11)

The reason for the involvement of the metric tensor is to preserve the scalar product:
\[ \forall \bar{a} : (\bar{v}, \bar{a}) = \bar{g}(\bar{v}, \bar{a}) = (\bar{g} \cdot \bar{v}) \cdot \bar{a} = v \cdot \bar{a} \]

The inverse operation “sharp” uses the inverse of the metric tensor:
\[ ^* : T^* M \rightarrow TM \]
\[ v \mapsto v^* = \bar{v} := \bar{g}^{-1} \cdot v \]  \hspace{1cm} (12)
Orthogonal Curvilinear Coordinate Systems

The components of tangent and cotangent vectors or tensors such as strain and stress tensors are often given as physical components (displacements in m, velocities in m/s, stress in Pa, etc.). These are then vector, covector etc. components using normalized basis vectors \( \{ e_i \} \), basis covectors \( \{ e^i \} \), basis two-forms \( \{ e^i \wedge e^j \} \) etc.). In contrast, it’s often convenient for calculations to use the canonical bases \( \{ \frac{\partial}{\partial x^i} \} \) and \( \{ dx^i \} \). Calculations in curvilinear coordinates involve swapping between these two types of base frequently.

Orthonormal bases are convenient when the Hodge star operator is applied [Fla90, Aga]:

\[
* e^i = \varepsilon_{ijk} e^j \wedge e^k \quad \text{or} \\
* (e^i \wedge e^j) = \varepsilon_{ijk} e^k \quad \text{(no summation, just pick one permutation of (1,2,3)).}
\]

\( \varepsilon_{ijk} = +1 \), if (i,j,k) is an even permutation of (1,2,3), or otherwise -1. However, this simple form is only valid for an orthonormal covector basis \( \{ e^i \} \), not generally for the canonical or other bases.

On the other hand, the exterior differential is most easily calculated in the canonical basis, e.g.:

\[
df = \frac{\partial f}{\partial x^i} dx^i \quad \text{or} \quad d\alpha = d(\alpha_i dx^i) = \frac{\partial \alpha_i}{\partial x^k} dx^k \wedge dx^i
\]

Since both operations are needed frequently, the need to change between orthonormal and canonical coordinates is obvious.

In the case of orthonormal coordinates\(^2\), there is a simple relationship between the two bases: \( e^i = h_i dx^i \) (no summation). The scaling factors \( h_i \) are most easily obtained by comparing the components of the riemannian metric tensor in both bases:

\[
\bar{g} = \delta_{ij} \otimes e^i \otimes e^j + e^2 \otimes e^2 + e^3 \otimes e^3 = \delta_{ij} e^i \otimes e^j \quad \text{in orthonormal and} \\
\tilde{g} = g_{ij} dx^i \otimes dx^j = h_1^2 dx^1 \otimes dx^1 + h_2^2 dx^2 \otimes dx^2 + h_3^2 dx^3 \otimes dx^3 \quad \text{in canonical bases.}
\]

Example: Cylindrical Coordinates

In cylindrical coordinates, for example, the metric tensor is \( \bar{g} = dr \otimes dr + r^2 d\theta \otimes d\theta + dz \otimes dz \). Therefore, \( h_r = h_z = 1 \) and \( h_\theta = r \). The basis covectors are \( e^r = dr \), \( e^\theta = r \, d\theta \) and \( e^z = dz \). The displacement form with physical components \((u_r, u_\theta, u_z)\) is then \( u = u_r e^r + u_\theta e^\theta + u_z e^z = u_r dr + u_\theta r d\theta + u_z dz \)

Kirchhoff Decomposition of the Displacement Field

It’s a well known fact, that a vector field can be decomposed in a scalar and a vector field:

\[
\bar{u} = \text{grad } \Phi + \text{rot } \Psi
\]

This form of decomposition is often used in electrostatics and –dynamics, as well as in the study of elastic waves. In differential form notation this can be written as:

\[
u = d\Phi + \delta \omega
\]

(10) then becomes:

\[
(\lambda + 2\mu) d\delta \omega + \mu \delta i d\delta u = (\lambda + 2\mu) d(\delta \Phi + \delta \omega) + \mu \delta i (d\Phi + \delta \omega)
\]

\[
= (\lambda + 2\mu) d^2 \delta \Phi + \mu \delta i d\delta \omega = 0
\]

since \( d^2 \delta = 0 \). As in vector notation, it can be shown that every displacement field, that satisfies (1) or (8) can be expressed as a sum of potentials, which seperately satisfy the equations \( d\delta \Phi = 0 \) and \( \delta i \delta \omega = 0 \).

**Conclusion**

This article showed how to formulate one of the most basic but also fundamental equations in elasticity theory – the displacement equation for linear isotropic threedimensional solids – using purely differential forms. Further...

\(^2\) A coordinate system is called orthogonal, if its canonical tangent space basis is orthogonal at each point.
thermore, it was briefly explained how to map between displacement vector fields and one-forms using the musical isomorphism. Kirchhoff vector field decomposition was

**Literature**


http://www.ee.byu.edu/forms/forms-home.html

[Bur] William L. Burke, various notes and publications on differential forms,
http://www.ee.byu.edu/forms/forms-home.html
